Exercise 8

The area of a triangle with sides of lengths a and b and contained angle θ is

$$A = \frac{1}{2}ab\sin\theta$$

(a) If $a = 2$ cm, $b = 3$ cm, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $\theta = \pi/3$?

(b) If $a = 2$ cm, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $b = 3$ cm and $\theta = \pi/3$?

(c) If a increases at a rate of 2.5 cm/min, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $a = 2$ cm, $b = 3$ cm, and $\theta = \pi/3$?

Solution

Part (a)

In this case, a and b are constant: $a = 2$ cm, $b = 3$ cm. Differentiate both sides of the formula for area with respect to t.

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\frac{1}{2}ab\sin\theta\right)$$

$$\frac{dA}{dt} = \frac{1}{2}ab\frac{d}{dt}(\sin\theta)$$

$$\frac{dA}{dt} = \frac{1}{2}ab(\cos\theta)\cdot\frac{d\theta}{dt}$$

θ increases at a rate of 0.2 rad/min, so $d\theta/dt = 0.2$ rad/min. Therefore, when $\theta = \pi/3$, the rate that the area increases is

$$\left.\frac{dA}{dt}\right|_{\theta=\pi/3} = \frac{1}{2}(2\text{ cm})(3\text{ cm})\left(\cos\frac{\pi}{3}\right)\cdot\left(0.2\,\frac{\text{rad}}{\text{min}}\right) = 0.3\,\frac{\text{cm}^2}{\text{min}}.$$

Part (b)

In this case, only a is constant: $a = 2$ cm. Differentiate both sides of the formula for area with respect to t.

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\frac{1}{2}ab\sin\theta\right)$$

$$\frac{dA}{dt} = \frac{1}{2}a\frac{d}{dt}(b\sin\theta)$$

$$\frac{dA}{dt} = \frac{1}{2}a\left\{\left[\frac{d}{dt}(b)\right]\sin\theta + b\left[\frac{d}{dt}(\sin\theta)\right]\right\}$$

$$\frac{dA}{dt} = \frac{1}{2}a\left\{\left[(1)\cdot\frac{db}{dt}\right]\sin\theta + b\left[(\cos\theta)\cdot\frac{d\theta}{dt}\right]\right\}$$

The rate of change for area is then

$$\frac{dA}{dt} = \frac{1}{2}a\left(\frac{db}{dt}\sin\theta + b\cos\theta\frac{d\theta}{dt}\right).$$

b increases at a rate of 1.5 cm/min, so $db/dt = 1.5$ cm/min. θ increases at a rate of 0.2 rad/min, so $d\theta/dt = 0.2$ rad/min. Therefore, when $b = 3$ cm and $\theta = \pi/3$, the rate that the area increases is

$$\begin{aligned}\left.\frac{dA}{dt}\right|_{\substack{b=3\\ \theta=\pi/3}} &= \frac{1}{2}(2\text{ cm})\left[\left(1.5\,\frac{\text{cm}}{\text{min}}\right)\sin\frac{\pi}{3} + (3\text{ cm})\left(\cos\frac{\pi}{3}\right)\left(0.2\,\frac{\text{rad}}{\text{min}}\right)\right]\\ &= \left(\frac{3}{2}\cdot\frac{\sqrt{3}}{2} + 3\cdot\frac{1}{2}\cdot\frac{1}{5}\right)\frac{\text{cm}^2}{\text{min}}\\ &= \frac{15\sqrt{3}+6}{20}\,\frac{\text{cm}^2}{\text{min}}.\end{aligned}$$

Part (c)

Differentiate both sides of the formula for area with respect to t.

$$\begin{aligned}\frac{d}{dt}(A) &= \frac{d}{dt}\left(\frac{1}{2}ab\sin\theta\right)\\ \frac{dA}{dt} &= \frac{1}{2}\frac{d}{dt}(ab\sin\theta)\\ \frac{dA}{dt} &= \frac{1}{2}\left\{\left[\frac{d}{dt}(a)\right]b\sin\theta + a\left[\frac{d}{dt}(b)\right]\sin\theta + ab\left[\frac{d}{dt}(\sin\theta)\right]\right\}\\ \frac{dA}{dt} &= \frac{1}{2}\left[\frac{da}{dt}b\sin\theta + a\frac{db}{dt}\sin\theta + ab(\cos\theta)\cdot\frac{d\theta}{dt}\right]\end{aligned}$$

a increases at a rate of 2.5 cm/min, so $da/dt = 2.5$ cm/min. b increases at a rate of 1.5 cm/min, so $db/dt = 1.5$ cm/min. θ increases at a rate of 0.2 rad/min, so $d\theta/dt = 0.2$ rad/min. Therefore, when $a = 2$ cm and $b = 3$ cm and $\theta = \pi/3$, the rate that the area increases is

$$\begin{aligned}\left.\frac{dA}{dt}\right|_{\substack{a=2\\ b=3\\ \theta=\pi/3}} &= \frac{1}{2}\left[\left(2.5\,\frac{\text{cm}}{\text{min}}\right)(3\text{ cm})\sin\frac{\pi}{3} + (2\text{ cm})\left(1.5\,\frac{\text{cm}}{\text{min}}\right)\sin\frac{\pi}{3} + (2\text{ cm})(3\text{ cm})\left(\cos\frac{\pi}{3}\right)\cdot\left(0.2\,\frac{\text{rad}}{\text{min}}\right)\right]\\ &= \frac{1}{2}\left(\frac{5}{2}\cdot 3\cdot\frac{\sqrt{3}}{2} + 2\cdot\frac{3}{2}\cdot\frac{\sqrt{3}}{2} + 2\cdot 3\cdot\frac{1}{2}\cdot\frac{1}{5}\right)\frac{\text{cm}^2}{\text{min}}\\ &= \frac{1}{2}\left(\frac{3}{5} + \frac{21\sqrt{3}}{4}\right)\frac{\text{cm}^2}{\text{min}}.\end{aligned}$$